Overview of course

Thomas Church
Stable homology through scanning
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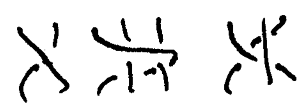
Homological stability is the phenomenon in low-dimensional topology:

		converges to:
$H_i(B_n) \approx H_i(B_{n+1})$	for $n \gg i$	$H_i(\Omega^2 S^2)$
$H_i(S_n) \approx H_i(S_{n+1})$	for $n \gg i$	$H_i(\Omega^\infty S^\infty)$
$H_i(\mathrm{Aut}(F_n)) \approx H_i(\mathrm{Aut}(F_{n+1}))$	for $n \gg i$	$H_i(\Omega^\infty S^\infty)$
$H_i(\mathrm{Mod}_g) \approx H_i(\mathrm{Mod}_{g+1})$	for $g \gg i$	$H_i(\Omega^\infty \mathbb{CP}^\infty_{-1})$ ← main goal of the course. answers the Mumford conjecture

But what does the homology converge to? What are these stable homology groups?

Scanning is a tool, used by Galatius but with precursors in earlier work, for answering this question. The key step is to recognize a natural geometric space whose homology we converge to.

Today we'll outline how to use scanning to prove that $H_i(B_n)$ converges to $H_i(\Omega^2_0 S^2)$ (the subscript means take only one component)

braid group B_n, elements look like 
but for our purposes, we only care that $B_n = \pi_1(X_n)$

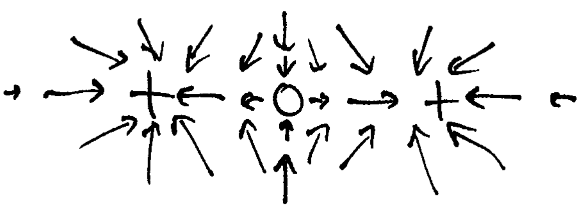
$X_n :=$ space of n distinct unordered points in the plane

$$= \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j\} / S_n$$

X_n is aspherical, so $H_i(B_n; \mathbb{Z}) = H_i(X_n; \mathbb{Z})$

$H_i(B_n;\mathbb{Z})$ converges to $H_i(\Omega^2_0 S^2;\mathbb{Z})$.
What is the relation?

X_n is the space of n distinct unordered points in the plane.
Put a +1eV charge at each point;
what is the force experienced by an electron?



$\left(F \sim \frac{1}{r^2}\right)$ goes to 0 at infinity
pole at each point

sending p to the vector F_p yields a map $\mathbb{R}^2 \to \mathbb{R}^2 \cup \{\infty\}$
invert through unit circle, get a map $\mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$ sending ∞ to ∞

$$X_n \longrightarrow \underset{\infty \mapsto \infty}{\mathrm{Maps}(S^2, S^2)} \longleftarrow \text{AKA } \Omega^2 S^2$$

It's ridiculous that $X_n \simeq \Omega^2 S^2$ — but let's try to prove it anyway.

- X_n is finite-dimensional, $\Omega^2 S^2$ is not
- X_n is aspherical with $\pi_1 = B_n$, $\Omega^2 S^2$ is not and has $\pi_1 = \mathbb{Z}$
- $X_n \not\simeq X_{n+1}$, so they can't be $\simeq$ to same space

Recall that B takes a topological group G and gives its classifying space BG (maps $X \to BG$ are same as G-bundles over X)

Claim: B is inverse to Ω.

Proof: $\pi_i(\Omega X) = \pi_{i+1}(X)$ ($\pi_0(\Omega X) = \pi_1(X)$ is definition of π_1; other cases same idea)

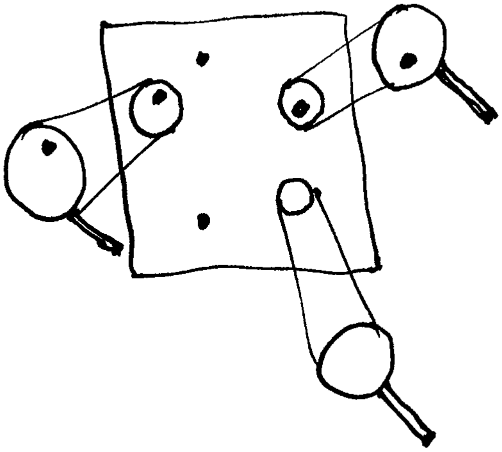
$\pi_i(G) = \pi_{i+1}(BG)$ ($\pi_{i+1}(BG)$ = maps $S^{i+1} \to BG$ = G-bundles over S^{i+1} determined by clutching function from equator $S^i \to G$)

so $G \simeq \Omega B G$ by Whitehead's theorem.

So to prove that $X_n \simeq \Omega^2 S^2$, it would suffice to prove that $BBX_n \simeq S^2$.

Same map $X_n \to \mathrm{Maps}(S^2, S^2)$, new perspective

Look through a microscope w/ autozoom
(field of vision $= \frac{1}{10} \cdot$ minimum distance between points)



At each point, we see through the viewfinder either one point, or nothing.
(the former becomes the latter as it slides out of view)
viewfinder/∂viewfinder $= S^2$

It turns out the important part is not zooming, so much as accepting that our vision is limited.

$Z :=$ space of finite subsets of $\square$,
topologized so points can slide out of frame continuously

$Z \overset{\text{zoom}}{\simeq} S^2$

$$\begin{tikzcd} & X_n \arrow[dl, "\substack{\text{new}\\ \text{camera}\\ \text{owner}}"'] \arrow[d, "\text{microscope}"] \\ \mathrm{Maps}(S^2, Z) \arrow[r, "\text{zoom}"'] & \mathrm{Maps}(S^2, S^2) \end{tikzcd}$$

new camera owner = take snapshots everywhere
(doesn't matter how much you zoom in or out)

Between X_n and Z we can interpolate

$Y :=$ space of finite subsets of $\square$,
topologized so points can slide out of frame at <u>top</u> and <u>bottom</u>

To talk about BX_n, we need a "multiplication". CONCATENATION:
but we need to take all n together

$X := \coprod X_n =$ space of finite subsets of $\square$

It turns out that Y really is BX, and Z really is BY!

so $BBX \simeq Z \simeq S^2$, and we get $X \simeq \Omega^2 S^2$?!?

<u>NO</u>. When we proved that $G \simeq \Omega BG$, it was only for topological <u>groups</u>.
Our spaces are only topological monoids, and it's impossible that

$$X \simeq \Omega BX = \Omega Y$$

(e.g. components of X have different fundamental groups $\pi_1 = B_n$)

Ex. $\mathbb{N}$ is a perfectly nice monoid.
What is π_1 of $B\mathbb{N} =$ "$K(\mathbb{N},1)$"? It's not $\mathbb{N}$, because π_1 is a group.
(It turns out $B\mathbb{N} = K(\mathbb{Z},1) = S^1$.)

It turns out that $G \simeq \Omega BG$ holds for any topological monoid where π_0 is a group.
(note that this implies all components of G are the same)

For other G, all we can say about G and ΩBG is:

Group Completion Theorem: $H_i(\Omega BG) = H_i(G)[\pi_0^{-1}]$

what does this mean?

G acts on itself by multiplication,
factors through action of $\pi_0 G$ on $H_i(G)$.
—FORCE this action to be invertible—

So: $H_i(X) = H_i(\amalg X_n) = \bigoplus H_i(X_n)$

$$H_i(X_0) \oplus H_i(X_1) \oplus H_i(X_2) \oplus H_i(X_3) \oplus H_i(X_4) \oplus \cdots \leftarrow$$ inverting action of π_0 is shifting the other way

$$H_i(X_0) \oplus H_i(X_1) \oplus H_i(X_2) \oplus H_i(X_3) \oplus \cdots$$

π_0 acts by shifting this over $\longrightarrow$ $$H_i(X_0) \oplus H_i(X_1) \oplus H_i(X_2) \oplus \cdots$$

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pretend there is a space X_∞ whose homology is the stable homology

$$H_i(X_\infty) = \lim_{n \to \infty} H_i(X_n)$$

$H_i(X)[\pi_0^{-1}]$:

$$\cdots \oplus H_i(X_\infty) \oplus H_i(X_\infty) \oplus H_i(X_\infty) \oplus H_i(X_\infty) \oplus H_i(X_\infty) \oplus \cdots = H_i(\mathbb{Z} \times X_\infty)$$

$$H_i(\mathbb{Z} \times X_\infty) = H_i(X)[\pi_0^{-1}] \overset{GCT}{=} H_i(\Omega BX) = H_i(\Omega Y) = H_i(\Omega\Omega BY) = H_i(\Omega\Omega Z) = H_i(\Omega^2 S^2)$$

$$\lim_{n \to \infty} H_i(B_n; \mathbb{Z}) = \lim_{n \to \infty} H_i(X_n; \mathbb{Z}) = H_i(\Omega_0^2 S^2)$$